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CERTAIN INTEGRAL TRANSFORMS AND THEIR APPLICATION TO GENERATE NEW LASER WAVES: EXTON-GAUSSIAN BEAMS

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Abstract. In this paper, a transformation is applied to Bessel-Circular-Gaussian beam to generate a new laser beam called Exton-Gaussian beams which have applications to wide areas of optics. For this, a novel integral transform involving the product of the Whittaker function and two Bessel functions of the first kind; the first with a linear argument and the second, which is a modified Bessel function, with a quadratic argument is derived whose solution is expressed in terms of the Exton's function X_8 and the Lauricella's function $F_A^{(3)}$. Some special cases of the main integral are illustrated by being specific on parameters. The analytical expression of the beam is utilized to study the propagation behavior. It is shown that the beam is composed of lobes and a dark center, potentially capable of trapping atoms in it. Such a behavior is important in physics and will be very useful for trapping atoms in its dark center and could be exploited as optical tweezers.

Keywords: Bessel function, Whittaker function, Lauricella's triple hypergeometric function, Exton's triple hypergeometric series.

AMS Subject Classification: 33B15, 33C10, 33C15.

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1 Introduction

Some mathematical research have been investigated during the last decades by evaluating some integrals involving special functions such as Bessel an Whittaker functions (Agarwal, 2013; Belafhal & Hennani, 2011; Becker, 2009; Choi & Agarwal, 2014; Khan et al., 2016). In the laser fields, these investigations are important to generate new beams waves, (Bandres & Vega, 2008; Mago et al., 2009; Salamin, 2018, 2019; Teng et al., 2018; Usman et al., 2018; Belafhal et al., 2020) and to study their propagation through mediums as free space, ABCD optical system, photonic crystals, chiral medium, turbulent and oceanic atmospheres, maritime turbulence and biological tissue (Andrews & Phillips, 2005; Belafhal & Dalil-Essakali, 2000; Dolev et al., 2009; Khannous et al., 2014; Khanous et al., 2018; Saad & Belafhal, 2017).

In the present work, we give a closed-form of the integral transform involving the product of the Whittaker function and two Bessel functions of the first kind; the first with a linear argument and the second, which is a modified Bessel function, with a quadratic argument. To the best of our knowledge, the evaluated closed-form of this integral transform is unprecedented and essential in the field of laser physics.

In the following, we give the definitions necessary for understanding the present investigation. For that, we start by the expansion of the Bessel function of the order ν given by (Watson, 1944)

$$J_\nu(z) = \sum_{m=0}^{\infty} \frac{(-1)^m (z/2)^{\nu+2m}}{m!\Gamma(\nu+m+1)}, \tag{1}$$

and the modified Bessel function given in terms of J_ν as

$$I_\nu(z) = i^{-\nu} J_\nu(iz). \tag{2}$$

The Appell and Lauricella's hypergeometric are defined, respectively, by (Srivastava & Manocha, 1984; Srivastava & Karlsson, 1985)

$$F_2\left[a, b_1, b_2; c_1, c_2; x, y\right] = \sum_{m,n=0}^{\infty} \frac{(a)_{m+n}(b_1)_m(b_2)_n}{(c_1)_m(c_2)_n} \frac{x^m}{m!} \frac{y^n}{n!}, \tag{3}$$

with $|x| + |y| < 1$, and

$$F_A^{(3)}\left[a, b_1, b_2, b_3; c_1, c_2, c_3; x, y, z\right] = \sum_{m,n,p=0}^{\infty} \frac{(a)_{m+n+p}(b_1)_m(b_2)_n(b_3)_p}{(c_1)_m(c_2)_n(c_3)_p} \frac{x^m}{m!} \frac{y^n}{n!} \frac{z^p}{p!}, \tag{4}$$

with $|x| + |y| + |z| < 1$.

Also,we recall the definition of the Humbert function defined as (Gradshteyn & Ryzhik, 1994; Srivastava & Karlsson, 1985)

$$\Psi_1\left[a, b; c_1, c_2; x, y\right] = \sum_{m,n=0}^{\infty} \frac{(a)_{m+n}(b)_m}{(c_1)_m(c_2)_n} \frac{x^m}{m!} \frac{y^n}{n!}, \tag{5}$$

with $|x| < 1$ and $|y| < \infty$.

The expression of the Exton's hypergeometric function as (Exton, 1982; Srivastava & Manocha, 1984)

$$X_8\left[a, b_1, b_2; c_1, c_2, c_3; x, y, z\right] = \sum_{m,n,p=0}^{\infty} \frac{(a)_{2m+n+p}(b_1)_n(b_2)_p}{(c_1)_m(c_2)_n(c_3)_p} \frac{x^m}{m!} \frac{y^n}{n!} \frac{z^n}{p!}, \tag{6}$$

with $2\sqrt{|x|} + |y| + |z| < 1$.

In the above functions, $(\lambda)_\nu$ is the shifted factorial (Srivastava & Karlsson, 1985) defined by

$$(\lambda)_\nu = \frac{\Gamma(\lambda+\nu)}{\Gamma(\lambda)} = \begin{cases} 1, & (\nu = 0, \lambda \in \mathbb{C} \setminus \{0\}) \\ \lambda(\lambda+1)\cdots(\lambda+n-1), & (\nu = n \in \mathbb{N}, \lambda \in \mathbb{C}) \end{cases}, \tag{7}$$

with a condition on each denominator of these equations that

$$c_j \in \mathbb{C} \setminus \mathbb{Z}_0^- \quad \text{(for j=1,2,3)}.$$

2 Main Result

We present in this section our main result derived with the help of the previous expressions and the relationship between the Exton's and Lauricella's functions elaborated recently by Choi & Rathie (2013).

Theorem 1. *The following integral transform holds true:*

$$\begin{aligned} I=&\int_0^\infty x^{2r}e^{-\alpha x^2}I_\chi(\lambda x^2)J_\mu(\beta x)M_{k,\nu}(2\gamma x^2)dx \\ =&A\ X_8\left[r+\varepsilon,\nu-k+\tfrac{1}{2},-;\chi+1,\mu+1,2\nu+1;x,y,z\right], \\ =&A\left(\frac{\alpha+\gamma}{\alpha+\gamma+\lambda}\right)^{r+\varepsilon}F_A^{(3)}\left[r+\varepsilon,\chi+\tfrac{1}{2},\nu-k+\tfrac{1}{2},-;\right. \\ &\qquad\qquad\left. 2\chi+1,\mu+1,2\nu+1;u,v,w\right], \end{aligned} \tag{8}$$

where

$$A=\sqrt{\frac{\gamma}{2}}\,\frac{\left(\frac{\beta}{2}\right)^\mu}{\mu!}\,\frac{\left(\frac{\lambda}{2}\right)^\chi}{\chi!}\,\frac{(2\gamma)^\nu\Gamma(r+\varepsilon)}{(\alpha+\gamma)^{r+\varepsilon}}, \tag{9}$$

$$\varepsilon=\chi+\frac{\mu}{2}+\nu+1, \tag{10}$$

$$x=\frac{\lambda^2}{4(\alpha+\gamma)^2},\ y=\frac{2\gamma}{\alpha+\gamma}\ \textit{and}\ z=\frac{-\beta^2}{4(\alpha+\gamma)}, \tag{11}$$

$$u=\frac{2\lambda}{\alpha+\gamma+\lambda},\ v=\frac{2\gamma}{\alpha+\gamma+\lambda}\ \textit{and}\ w=\frac{-\beta^2}{4(\alpha+\gamma+\lambda)}, \tag{12}$$

with

$$|x|+|y|+|z|<1, \textit{and}\ 2\sqrt{|u|}+|v|+|w|<1.$$

Proof. By using the expansion of the modified Bessel function given by (2), I becomes

$$I=\sum_{m=0}^\infty\frac{(\lambda/2)^{\chi+2m}}{m!\Gamma(\chi+m+1)}A_m, \tag{13}$$

where

$$A_m=\int_0^\infty x^{2s}e^{-\alpha x^2}J_\mu(\beta x)M_{k,\nu}\left(2\gamma x^2\right)dx, \tag{14}$$

with

$$s=2m+r+\chi. \tag{15}$$

With the help of Theorem 1 of Khan et al. (2016), (14) can be written as

$$\begin{aligned} A_m=&\frac{\beta^\mu\gamma^{\nu+\frac{1}{2}}}{\mu!2^{\mu-\nu+\frac{1}{2}}}\frac{\Gamma(2m+r+\varepsilon)}{(\alpha+\gamma)^{2m+r+\varepsilon}} \\ &\times\Psi_1\left[2m+r+\varepsilon,\nu-k+\frac{1}{2};\mu+1,2\nu+1;\frac{2\gamma}{\alpha+\gamma},\frac{-\beta^2}{4(\alpha+\gamma)}\right]. \end{aligned} \tag{16}$$

Consequently, if one uses the identity Srivastava & Karlsson (1985)

$$\Gamma(r+\varepsilon)(r+\varepsilon)_{2m+n+p}=(2m+r+\varepsilon)_{n+p}\Gamma(2m+r+\varepsilon), \tag{17}$$

(13) can be rearranged as

$$I=A\sum_{m,p,n=0}^\infty\frac{(r+\varepsilon)_{2m+n+p}(\nu-k+\frac{1}{2})_m}{(\chi+1)_m(\mu+1)_n(2\nu+1)_p}\frac{x^m}{m!}\frac{y^p}{p!}\frac{z^n}{n!}, \tag{18}$$

where, x, y and z are given by (11).
Thus, by using the Exton's function X_8 given by (6), (8) is proved.

We express the integral I in terms of Lauricella's triple hypergeometric function given by (8). For that, by using the following relation Choi & Rathie (2013)

$$\begin{aligned} & X_8\left[a, b_1, b_2; c_1+\frac{1}{2}, c_2, c_3; x^2, y, z\right] \\ & =\frac{1}{(1+2x)^a}\ F_A^{(3)}\left[a, c_1, b_1, b_2; 2c_1, c_2, c_3; \frac{4x}{1+2x}, \frac{y}{1+2x}, \frac{z}{1+2x}\right], \end{aligned} \tag{19}$$

and by taking

$$\begin{aligned} a &= r+\varepsilon, \\ b_1 &= \nu-k+\frac{1}{2}, \\ b_2 &= 0, \\ c_1 &= \chi+\frac{1}{2}, \\ c_2 &= \mu+1, \end{aligned}$$

and

$$c_3 = 2\nu+1,$$

one finds (8) and this completes the proof. □

3 Special Cases

In this section, we investigate some special cases of the main result (8) corresponding to some values of parameters of Bessel and Whittaker functions.

Corollary 1. *The following integral transform holds true:*

$$\begin{aligned} & \int_0^{\infty} x^{2(r-k)} e^{-(\alpha-\gamma)x^2} I_\chi(\lambda x^2) J_\mu(\beta x) dx\ = A(2\gamma)^k \\ & \times X_8\left[r+\varepsilon, -2k, -; \chi+1, \mu+1, 2k; \frac{\lambda^2}{4(\alpha+\gamma)^2}, \frac{2\gamma}{\alpha+\gamma}, \frac{-\beta^2}{4(\alpha+\gamma)}\right], \\ & = A(2\gamma)^k\left(\frac{\alpha+\gamma}{\alpha+\gamma+\lambda}\right)^{r+\varepsilon} F_A^{(3)}\left[r+\varepsilon, \chi+\frac{1}{2}, -2k, -; 2\chi+1, \mu+1, 2k;\right. \\ & \left.\frac{2\lambda}{\alpha+\gamma+\lambda}, \frac{2\gamma}{\alpha+\gamma+\lambda}, \frac{-\beta^2}{4(\alpha+\gamma+\lambda)}\right], \end{aligned} \tag{20}$$

where

$$\varepsilon = \chi+\frac{\mu+1}{2}-k.$$

Proof. By taking $\nu = -(k+\frac{1}{2})$, the Whittaker function is given by (Srivastava & Manocha, 1984)

$$M_{k,-k-\frac{1}{2}}(z) = e^{\frac{z}{2}} z^{-k}, \tag{21}$$

and from (8) one finds (20). This complets the proof of the corollary. □

Corollary 2. *The following integral transform holds true:*

$$\begin{aligned}\int_0^\infty x^{2r+1}e^{-\alpha x^2}I_\chi(\lambda x^2)J_\mu(\beta x)I_\nu(\gamma x^2)dx \ &= \frac{B}{(\alpha+\gamma)^{r+\varepsilon}}\\ \times X_8\left[r+\varepsilon,\nu+\frac{1}{2},-;\chi+1,\mu+1,2\nu+1;\frac{\lambda^2}{4(\alpha+\gamma)^2},\frac{2\gamma}{\alpha+\gamma},\frac{-\beta^2}{4(\alpha+\gamma)}\right],\\ =&\frac{B}{(\alpha+\gamma+\lambda)^{r+\varepsilon}}F_A^{(3)}\left[r+\varepsilon,\chi+\frac{1}{2},\nu+\frac{1}{2},-;2\chi+1,\mu+1,2\nu+1;\right.\\ &\left.\frac{2\lambda}{\alpha+\gamma+\lambda},\frac{2\gamma}{\alpha+\gamma+\lambda},\frac{-\beta^2}{4(\alpha+\gamma+\lambda)}\right],\end{aligned} \tag{22}$$

where

$$B=\frac{(\frac{\lambda}{2})^\chi}{\chi!}\frac{(\frac{\beta}{2})^\mu}{\mu!}\frac{(\frac{\gamma}{2})^\nu}{\nu!}\frac{\Gamma(r+\varepsilon)}{2},$$

and

$$\varepsilon=\chi+\frac{\mu}{2}+\nu+1.$$

Proof. If we take $k=0$, the Whittaker function becomes (Srivastava & Manocha, 1984)

$$M_{0,\nu}(2z)=2^{2\nu+\frac{1}{2}}\nu!\sqrt{z}I_\nu(z), \tag{23}$$

and it's easy to find (22) from (8). □

Corollary 3. *The following integral transform holds true:*

$$\begin{aligned}&\int_0^\infty x^{2r}e^{-\alpha x^2}I_\chi(\lambda x^2)J_\mu(\beta x)sinh(\gamma x^2)dx\\ &=CX_8\left[r+\varepsilon,1,-;\chi+1,\mu+1,2;\frac{\lambda^2}{4(\alpha+\gamma)^2},\frac{2\gamma}{\alpha+\gamma},\frac{-\beta^2}{4(\alpha+\gamma)}\right],\\ &=C\left(\frac{\alpha+\gamma}{\alpha+\gamma+\lambda}\right)^{r+\varepsilon}F_A^{(3)}\left[r+\varepsilon,\chi+\frac{1}{2},1,-;2\chi+1,\mu+1,2;\right.\\ &\left.\frac{2\lambda}{\alpha+\gamma+\lambda},\frac{2\gamma}{\alpha+\gamma+\lambda},\frac{-\beta^2}{4(\alpha+\gamma+\lambda)}\right],\end{aligned} \tag{24}$$

where

$$C=\frac{\gamma}{2}\frac{(\frac{\lambda}{2})^\chi}{\chi!}\frac{(\frac{\beta}{2})^\mu}{\mu!}\frac{\Gamma(r+\varepsilon)}{(\alpha+\gamma)^{r+\varepsilon}},$$

and

$$\varepsilon=\chi+\frac{\mu}{2}+\frac{3}{2}.$$

Proof. It's easy, by taking $k=0$ and $\nu=\frac{1}{2}$, to use (8) with (Srivastava & Manocha, 1984)

$$M_{0,\frac{1}{2}}(z)=2\sinh\left(\frac{z}{2}\right), \tag{25}$$

to prove (24). □

Corollary 4. *The following integral transform holds true:*

$$\begin{aligned}&\int_0^\infty x^{2r+p+1}e^{-(\alpha+\gamma)x^2}I_\chi(\lambda x^2)J_\mu(\beta x)L_s^p(2\gamma x^2)dx\\ &=DX_8\left[r+\varepsilon,-s,-;\chi+1,\mu+1,p+1;\frac{\lambda^2}{4(\alpha+\gamma)^2},\frac{2\gamma}{\alpha+\gamma},\frac{-\beta^2}{4(\alpha+\gamma)}\right],\\ &=D\left(\frac{\alpha+\gamma}{\alpha+\gamma+\lambda}\right)^{r+\varepsilon}F_A^{(3)}\left[r+\varepsilon,\chi+\frac{1}{2},-s,-;2\chi+1,\mu+1,p+1;\right.\\ &\left.\frac{2\lambda}{\alpha+\gamma+\lambda},\frac{2\gamma}{\alpha+\gamma+\lambda},\frac{-\beta^2}{4(\alpha+\gamma+\lambda)}\right],\end{aligned} \tag{26}$$

where

$$D = \frac{(p+1)_s}{2s!} \frac{(\frac{\lambda}{2})^\chi}{\chi!} \frac{(\frac{\beta}{2})^\mu}{\mu!} \frac{\Gamma(r+\varepsilon)}{(\alpha+\gamma)^{r+\varepsilon}},$$

and

$$\varepsilon = \chi + \frac{\mu+p}{2} + 1.$$

Proof. If we take $k = \frac{p+1}{2} + s$ and $\nu = \frac{p}{2}$, the Whittaker function is expressed as (Srivastava & Manocha, 1984)

$$M_{\frac{p+1}{2}+s,\frac{p}{2}}(z) = \frac{s!}{(p+1)_s} e^{\frac{-z}{2}} z^{\frac{p+1}{2}} L_s^{(p)}(z), \tag{27}$$

where $L_s^{(p)}$ is the generalized Laguerre polynomial. So, with the help of (8) the corollary is proved. □

Corollary 5. *The following integral transform holds true:*

$$\begin{aligned}
&\int_0^\infty x^{2r+\frac{1}{2}} e^{-(\alpha+\gamma)x^2} I_\chi(\lambda x^2) J_\mu(\beta x) H_{2p}(\sqrt{2\gamma}x) dx \\
&= \frac{E}{(\alpha+\gamma)^{r+\varepsilon}} X_8 \left[r+\varepsilon, -p, -; \chi+1, \mu+1, \frac{1}{2}; \frac{\lambda^2}{4(\alpha+\gamma)^2}, \frac{2\gamma}{\alpha+\gamma}, \frac{-\beta^2}{4(\alpha+\gamma)} \right], \\
&= \frac{E}{(\alpha+\gamma+\lambda)^{r+\varepsilon}} F_A^{(3)} \left[r+\varepsilon, \chi+\frac{1}{2}, -p, -; \chi+1, \mu+1, \frac{1}{2}; \right. \\
&\qquad \left. \frac{2\lambda}{\alpha+\gamma+\lambda}, \frac{2\gamma}{\alpha+\gamma+\lambda}, \frac{-\beta^2}{4(\alpha+\gamma+\lambda)} \right],
\end{aligned} \tag{28}$$

where

$$E = \frac{(2p)!}{2p!(-1)^p} \frac{(\frac{\lambda}{2})^\chi}{\chi!} \frac{(\frac{\beta}{2})^\mu}{\mu!} \Gamma(r+\varepsilon),$$

and

$$\varepsilon = \chi + \frac{\mu}{2} + \frac{3}{4}.$$

Proof. In this case, we have an expression of the Whittaker function in terms of the Hermite polynomial. The above corollary can easily be established by setting $k = p + \frac{1}{4}$ and $\nu = \frac{-1}{4}$ with the odd integer order and using the relation betweeen M and H_{2p} given by (Srivastava & Manocha, 1984)

$$M_{p+\frac{1}{4},-\frac{1}{4}}\left(z^2\right) = (-1)^p \frac{p!}{(2p)!} e^{\frac{-z^2}{2}} \sqrt{z} H_{2p}(z), \tag{29}$$

we deduce (28) from (8). □

Corollary 6. *The following integral transform holds true:*

$$\begin{aligned}
&\int_0^\infty x^{2r+\frac{1}{2}} e^{-(\alpha+\gamma)x^2} I_\chi(\lambda x^2) J_\mu(\beta x) H_{2p+1}(\sqrt{2\gamma}x) dx \\
&= \frac{F}{(\alpha+\gamma)^{r+\varepsilon}} X_8 \left[, r+\varepsilon, -p, -; \chi+1, \mu+1, \frac{3}{2}; \frac{\lambda^2}{4(\alpha+\gamma)^2}, \frac{2\gamma}{\alpha+\gamma}, \frac{-\beta^2}{4(\alpha+\gamma)} \right], \\
&= \frac{F}{(\alpha+\gamma+\lambda)^{r+\varepsilon}} F_A^{(3)} \left[r+\varepsilon, \chi+\frac{1}{2}, -p, -; 2\chi+1, \mu+1, \frac{3}{2}; \right. \\
&\qquad \left. \frac{2\lambda}{\alpha+\gamma+\lambda}, \frac{2\gamma}{\alpha+\gamma+\lambda}, \frac{-\beta^2}{4(\alpha+\gamma+\lambda)} \right],
\end{aligned} \tag{30}$$

where

$$F=\frac{\sqrt{\gamma}}{2}\frac{\left(\frac{\lambda}{2}\right)^{\chi}}{\chi!}\frac{\left(\frac{\beta}{2}\right)^{\mu}}{\mu!}\frac{(2p+1)!}{(-1)^{p}p!}\Gamma(r+\varepsilon),$$

and

$$\varepsilon=\chi+\frac{\mu}{2}+\frac{5}{4}.$$

Proof. By setting $k=p+\frac{3}{4}$ and $\nu=\frac{1}{4}$, this corollary is established by using Theorem 1 and the relation between the Whittaker function and the Hermite polynomial H_{2p+1} of an even order given by (Srivastava & Manocha, 1984)

$$M_{p+\frac{3}{4},\frac{1}{4}}\left(z^{2}\right)=(-1)^{p}\frac{p!}{(2p+1)!}\frac{e^{\frac{-z^{2}}{2}}\sqrt{z}}{2}H_{2p+1}(z). \tag{31}$$

□

Corollary 7. *The following integral transform holds true:*

$$\begin{aligned}
&\int_{0}^{\infty}x^{2r-2\mu+1}e^{-(\alpha-\varepsilon)x^{2}}I_{\chi}(\lambda x^{2})J_{\mu}(\beta x)\gamma(2\mu,2\xi x^{2})dx\\
&=\frac{G}{(\alpha+\gamma)^{r+\varepsilon}}X_{8}\left[r+\varepsilon,1,-;\chi+1,\mu+1,2\mu+1;\frac{\lambda^{2}}{4(\alpha+\gamma)^{2}},\frac{2\gamma}{\alpha+\gamma},\frac{-\beta^{2}}{4(\alpha+\gamma)}\right],\\
&=\frac{G}{(\alpha+\gamma+\lambda)^{r+\varepsilon}}F_{A}^{(3)}\left[r+\varepsilon,\chi+\frac{1}{2},1,-;2\chi+1,\mu+1,2\mu+1;\right.\\
&\qquad\left.\frac{2\lambda}{\alpha+\gamma+\lambda},\frac{2\gamma}{\alpha+\gamma+\lambda},\frac{-\beta^{2}}{4(\alpha+\gamma+\lambda)}\right],
\end{aligned} \tag{32}$$

where

$$G=\frac{(2\beta\xi^{2})^{\mu}}{4\mu\mu!}\frac{\left(\frac{\lambda}{2}\right)^{\chi}}{\chi!}\Gamma(r+\varepsilon),$$

and

$$\varepsilon=\chi+\frac{3\mu}{2}+1.$$

Proof. This corollary is proved by taking $k=\mu-\frac{1}{2}$ and $\nu=\mu$. In this case we have (Srivastava & Manocha, 1984)

$$M_{\mu-\frac{1}{2},\mu}\left(z\right)=2\ \mu\ e^{\frac{z}{2}}z^{\frac{1}{2}-\mu}\gamma(2\ \mu,z), \tag{33}$$

and with the help of Theorem 1, one finds (32). □

Corollary 8. *The following integral transform holds true:*

$$\begin{aligned}
&\int_{0}^{\infty}x^{2r+\frac{1}{2}}e^{-(\alpha-\gamma)x^{2}}I_{\chi}(\lambda x^{2})J_{\mu}(\beta x)erf(\sqrt{2\gamma}x)dx\\
&=\frac{H}{(\alpha+\gamma)^{r+\varepsilon}}X_{8}\left[r+\varepsilon,1,-;\chi+1,\mu+1,\frac{3}{2};\frac{\lambda^{2}}{4(\alpha+\gamma)^{2}},\frac{2\gamma}{\alpha+\gamma},\frac{-\beta^{2}}{4(\alpha+\gamma)}\right],\\
&=\frac{H}{(\alpha+\gamma+\lambda)^{r+\varepsilon}}F_{A}^{(3)}\left[r+\varepsilon,\chi+\frac{1}{2},1,-;2\chi+1,\mu+1,\frac{3}{2};\right.\\
&\qquad\left.\frac{2\lambda}{\alpha+\gamma+\lambda},\frac{2\gamma}{\alpha+\gamma+\lambda},\frac{-\beta^{2}}{4(\alpha+\gamma+\lambda)}\right],
\end{aligned} \tag{34}$$

where

$$H=\sqrt{\frac{2\gamma}{\pi}}\frac{\left(\frac{\lambda}{2}\right)^{\chi}}{\chi!}\frac{\left(\frac{\beta}{2}\right)^{\mu}}{\mu!}\Gamma(r+\varepsilon),$$

and

$$\varepsilon=\chi+\frac{\mu}{2}+\frac{5}{4}.$$

Proof. By taking $k = \frac{-1}{4}$ and $\nu = \frac{1}{4}$, the Whittaker function is reduced in terms of the erf function as (Srivastava & Manocha, 1984)

$$M_{\frac{-1}{4},\frac{1}{4}}\left(z^2\right) = \frac{1}{2}e^{\frac{z^2}{2}}\sqrt{\pi z}\; erf(z), \tag{35}$$

from which it is easy to deduce Corrollary 8. □

Corollary 9. *The following integral transforms hold true:*

$$\begin{aligned}
&\int_0^\infty x^{2r+\frac{1}{2}}e^{-\alpha x^2}I_\chi(\lambda x^2)J_\mu(\beta x)\left[U(p,-2\sqrt{\gamma}x)-U(p,2\sqrt{\gamma}x)\right]dx\\
&= \frac{J}{(\alpha+\gamma)^{r+\varepsilon}}X_8\left[r+\varepsilon,\frac{p}{2}+\frac{3}{4},-;\chi+1,\mu+1,\frac{3}{2};\frac{\lambda^2}{4(\alpha+\gamma)^2},\frac{2\gamma}{\alpha+\gamma},\frac{-\beta^2}{4(\alpha+\gamma)}\right],\\
&= \frac{J}{(\alpha+\gamma+\lambda)^{r+\varepsilon}}F_A^{(3)}\left[r+\varepsilon,\chi+\frac{1}{2},\frac{p}{2}+\frac{3}{4},-;2\chi+1,\mu+1,\frac{3}{2};\right.\\
&\qquad\left.\frac{2\lambda}{\alpha+\gamma+\lambda},\frac{2\gamma}{\alpha+\gamma+\lambda},\frac{-\beta^2}{4(\alpha+\gamma+\lambda)}\right],
\end{aligned} \tag{36}$$

where

$$J = \frac{4\sqrt{\pi\gamma}}{2^{\frac{P+3}{4}}\Gamma(\frac{P}{2}+\frac{1}{4})}\frac{(\frac{\lambda}{2})^\chi}{\chi!}\frac{(\frac{\beta}{2})^\mu}{\mu!}\Gamma(r+\varepsilon),$$

and

$$\varepsilon = \chi + \frac{\mu}{2} + \frac{5}{4}.$$

Proof. This corollary is established by setting $k = \frac{-p}{2}$ and $\nu = \frac{1}{4}$ and Theorem 1. In this case, the relation between the Whittaker function and the Parabolic Cylinder function U is given as follows (Srivastava & Manocha, 1984)

$$M_{\frac{-p}{2},\frac{1}{4}}\left(\frac{z^2}{2}\right) = \frac{2^{\frac{p}{2}}}{4}\Gamma\left(\frac{p}{2}+\frac{1}{4}\right)\sqrt{\frac{z}{\pi}}\;\left\{U(p,-z)-U(p,z)\right\}. \tag{37}$$

□

4 Application: Generation of Exton-Gaussian Beams

In this section, we transform a Bessel-circular-Gaussian beam (BCiGBs) by a spiral phase plate (SPP) (see Fig.1). This system will generate a new donut beam as an azimuth dependent phase retardation is imposed by the SPP. This phase is given by $e^{i\chi\phi'}$ where χ is the topological charge of the SPP (Belafhal & Saad, 2017).

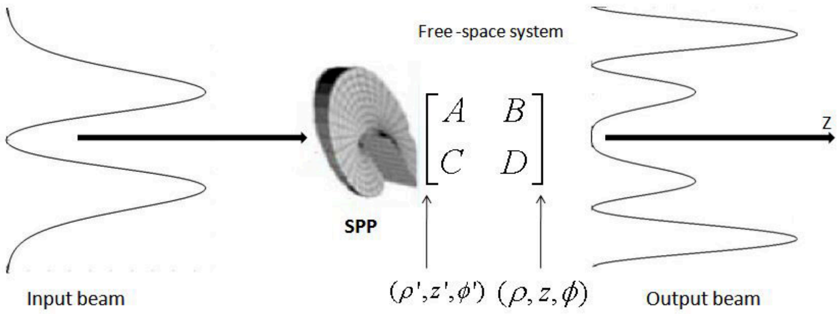

Figure 1: Illustration of the propagation of BCiGBS by a SPP through an ABCD optical system

After the SPP, the input field can be written as

$$E_{\mu,m}(\rho',z',\phi') = Fe^{i(m+\chi)\phi'}\frac{e^{ikz'}}{\rho'}M_{\frac{\mu}{2},\frac{m}{2}}\left(\frac{i\omega_0{}^2}{2\chi'^2}\rho'^2\right)I_\nu(\varepsilon\rho'^2)e^{\frac{ik\omega_0{}^2}{4}\left(\frac{1}{q'(z)}+\frac{1}{\widetilde{q}\,'(z)}\right)\rho'^2}, \tag{38}$$

where

$$F = E_f\left(\frac{1+(z'/\widetilde{q}_0)}{1+(z'/q_0)}\right)^{\frac{\mu}{2}}\left(\frac{2\chi'^2}{i\omega_0{}^2}\right)^{\frac{1}{2}}\frac{1}{[1+(z'/q_0)]},$$
$$q' = z'+q_0,$$
$$\widetilde{q}' = z'+\widetilde{q}_0,$$

and

$$\frac{1}{\chi'^2} = k\left(\frac{1}{\widetilde{q}\,'}-\frac{1}{q'}\right),$$

with (ρ',z',ϕ') are the cylindrical coordinates and $(q_0,\widetilde{q}_0)$ are parameters of the fundamental Gaussian envelope.

By applying the Collins-Huygens integral (Collins, 1970) on this beam propagating through an ABCD optical system (see Fig.1), one finds the following output field expressed, for χ equal to an integer l, as

$$E_{\mu,m}(\rho,z,\phi) = \frac{-iF}{\lambda B}e^{ik\left(z+\frac{D\rho^2}{2B}\right)}e^{ikz'}I^l_{\mu,m}(\rho), \tag{39}$$

where

$$\begin{aligned}I^l_{\mu,m}(\rho) = \int_0^\infty\int_0^{2\pi} & e^{-\alpha\rho'^2}e^{\frac{-ik\rho\rho'}{B}\cos(\phi-\phi')}M_{\frac{\mu}{2},\frac{m}{2}}\left(\frac{i\omega_0{}^2}{2\chi'^2}\rho'^2\right)\\ & \times I_\nu(\varepsilon\rho'^2)e^{i(m+l)\phi'}d\rho' d\phi',\end{aligned} \tag{40}$$

with

$$\alpha = -ik\left[\frac{\omega_0{}^2}{4}\left(\frac{1}{q'(z)}+\frac{1}{\widetilde{q}\,'(z)}\right)+\frac{A}{2B}\right]. \tag{41}$$

With the help of the following identity (Abramowitz & Stegun, 1970)

$$\int_0^{2\pi} e^{in\phi'}e^{\frac{-ik\rho\rho'}{B}\cos(\phi-\phi')}d\phi' = 2\pi i^n e^{in\phi}J_n\left(\frac{k\rho\rho'}{B}\right), \tag{42}$$

(40) can be written as

$$I^l_{\mu,m}(\rho) = 2\pi i^{(m+l)}e^{i(m+l)\phi}I, \tag{43}$$

where

$$I = \int_0^\infty e^{-\alpha\rho'^2}I_\nu(\varepsilon\rho'^2)M_{\frac{\mu}{2},\frac{m}{2}}\left(\frac{i\omega_0^2}{2\chi'^2}\rho'^2\right)J_{m+l}\left(\frac{k\rho\rho'}{B}\right)d\rho'. \tag{44}$$

By using Theorem 1, (39) can be expressed as

$$\begin{aligned}E_{\mu,m}(\rho,z,\phi) = & \frac{-2\pi}{\lambda B}e^{ik\left(z+z'+\frac{D\rho^2}{2B}\right)}i^{(m+l+1)}e^{i(m+l)\phi}.F.G\\ & \times X_8\left[\nu+m+\frac{l}{2}+1,\frac{m-\mu+1}{2},-;\nu+1,m+l+1,m+1;x,y,z\right],\end{aligned} \tag{45}$$

where

$$G=\sqrt{\frac{i\omega_0^2}{8\chi'^2}}\,\frac{\left(\frac{k\rho}{2B}\right)^{m+l}}{(m+l)!}\,\frac{\left(\frac{\varepsilon}{2}\right)^{\nu}}{\nu!}\,\frac{\left(\frac{i\omega_0^2}{2\chi'^2}\right)^{\frac{m}{2}}\Gamma\left(\nu+m+\frac{l}{2}+1\right)}{\left(\alpha+\frac{i\omega_0^2}{4\chi'^2}\right)^{\nu+m+\frac{l}{2}+1}},$$

$$x=\frac{\varepsilon^2}{4\left(\alpha+\frac{i\omega_0^2}{4\chi'^2}\right)^2},\ y=\frac{\frac{i\omega_0^2}{2\chi'^2}}{\alpha+\frac{i\omega_0^2}{4\chi'^2}}\ \textit{and}\ z=-\frac{(k\rho)^2}{4B^2\left(\alpha+\frac{i\omega_0^2}{4\chi'^2}\right)},$$

and

$$u=\frac{2\varepsilon}{\alpha+\frac{i\omega_0^2}{4\chi'^2}+\varepsilon},\ v=\frac{\frac{i\omega_0^2}{2\chi'^2}}{\alpha+\frac{i\omega_0^2}{4\chi'^2}+\varepsilon},\ \textit{and}\ w=-\frac{(k\rho)^2}{4B^2\left(\alpha+\frac{i\omega_0^2}{4\chi'^2}+\varepsilon\right)}.$$

Finally, our main result permits us to generate new laser waves called the Exton-Gaussian beams defined by (45).

Now, we present some numerical simulations to study the dependence of the propagation of this new beam on the propagation distance z, the radial coordinate ρ and the beam orders. The present calculations are based on the complex amplitude expression given by (45). From this last quantity, we deduce the intensity distribution of our donut beam which is evaluated by $I(\rho,z,\phi)=|E(\rho,z,\phi)|^2$ at any distance and for any beam orders behind the SPP.

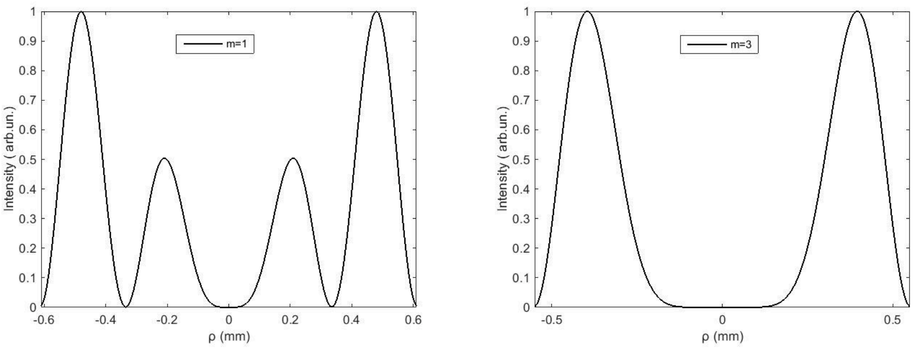

Figure 2: Normalized intensity distribution of the Exton-Gaussian beam with $l=1$ and for two values of m

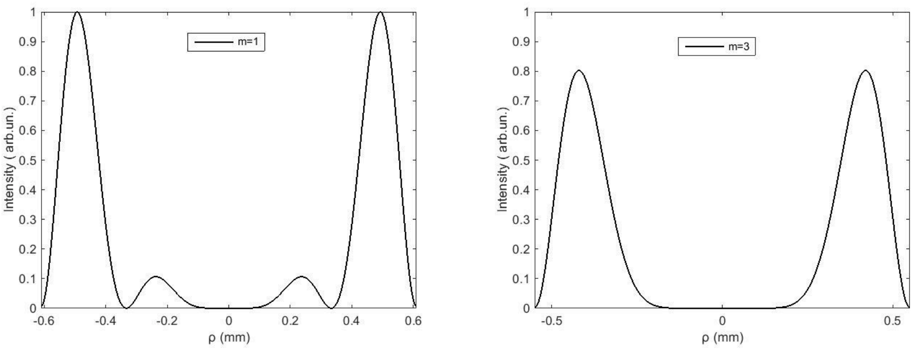

Figure 3: Normalized intensity distribution of the Exton-Gaussian beam with $l=2$ and for two values of m

So, for our illustration, we consider the propagation of the beam in a free space defined by a transfer matrix ABCD where the elements are: $A=1, B=z, C=0, \textit{and}\ D=1$. The beam parameters are chosen as: the beam waist $\omega_0 = 0.6\,mm$ and the wavelength $\lambda = 1060\,nm$. We illustrate in Figs. 2 and 3, the normalized intensity distribution of the Exton-Gaussian beam at the propagation distance $z = 4m$ with a topological charge $l = 1$ and 2, respectively.

For different values of beam orders (see legends of these figures) and as predicted by (45), the beam intensity is composed by some lobes, and the beam seems with a dark centre (the intensity is null in this region). This property is very important in physics and will be very useful for trapping atoms in its dark centre and could be exploited as optical tweezers.

5 Conclusion

In this paper, we have investigated an integral transforms involving the product of Bessel and modified Bessel functions and Whittaker function. The result is derived in terms of the Exton's function X_8 and the Lauricella's hypergeometric function $F_A^{(3)}$. Some special cases are examined for special values of the parameters of the Whittaker function. At the end of this investigation and as application, we have investigated in section 4, the possibility of generating new donut waves called Exton-Gaussian beams.

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